MATH 590: QUIZ 11

Name:

1. For an $n \times n$ matrix A defined over $\mathbb{C}$, define the adjoint of A and use this definition to state what it means for A to be unitary (complex orthogonal). (4 points)

Solution. The adjoint A^* of A is defined as $A^* := (\overline{A})^t = \overline{A^t}$. A is unitary if $A^{-1} = A^*$.

2. Show that the matrix $A = \begin{pmatrix} 3 & 2i \\ -2i & 3 \end{pmatrix}$ is normal and then find a unitary matrix P such that P^*AP is a diagonal matrix. Be sure to justify that P is unitary. Note: You just have to find the correct P, you do not have to check P^*AP is diagonal. (6 points)

Solution. $p_A(x) = \begin{vmatrix} x-3 & -2i \\ 2i & x-3 \end{vmatrix} = (x-3)^2 + (2i)^2 = (x^2 - 6x + 9) - 4 = x^2 - 6x + 5 = (x-5)(x-1)$, so the eigenvalues of A are 5 and 1.

$E_5 =$ null space of $\begin{pmatrix} -2 & 2i \\ -2i & -2 \end{pmatrix} \overset{\text{EROs}}{\rightarrow} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$, so that $u_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ is a unit length eigenvector for 5.

$E_1 =$ null space of $\begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} \overset{\text{EROs}}{\rightarrow} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$, so that $u_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is a unit length eigenvector for 1.

Note that $u_1 \cdot u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{2} \cdot (i \cdot \overline{(-i)} + 1 \cdot \overline{1}) = \frac{1}{2}(-1+1) = 0$, so that u_1 and u_2 are orthogonal.

Since the unit length eigenvectors are orthogonal, the matrix $P = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is a unitary matrix that diagonalizes A.